New embeddings for nonlinear multiobjective optimization problems I

Jürgen GUDDAT
Humboldt-Universität zu Berlin
Institut für Mathematik

Francisco GUERRA
Universidad de las Americas, Puebla
Departamento Fisica y Matematicas

Dieter NOWACK
Humboldt-Universität zu Berlin
Institut für Mathematik

February 12, 1998

Abstract

In a dialogue procedure the decision maker has to determine in each step the aspiration and reservation level expressing his wishes (goals). This leads to an optimization problem which is not easy to solve in the nonconvex case (the known starting point is not feasible). We propose a modified standard embedding (one parametric optimization). This problem will be discussed from the point of view of parametric optimization (non-degenerate critical points, singularities, pathfollowing methods to describe numerically a connected component in the set of stationary points and in the set of generalized critical points, respectively, and jumps (descent methods) to other connected components in these sets). This embedding is much better for computing a goal realizer or replying that the goal was not realistic than the embeddings considered in the literature before, but in the worst case we have to find all connected components and this is an open problem.

Keywords: Multiobjective optimization, non-degenerate critical points, singularities, pathfollowing methods

1 Introduction

We consider the following multiobjective optimization problem

$$(MOP) \quad \min\{(f_1(x), \dots, f_l(x)) \mid x \in M\}$$

where

$$\begin{aligned} M :=& \quad \{x \in I\!R^n \mid g_j(x) \le 0, j \in J\} \\ J :=& \quad \{1, \dots, s\}, K := \{1, \dots, l\},\ f_k, g_j \in C^q(I\!R^n, I\!R), k \in K, j \in J, q \in \{2, 3\} \end{aligned}$$

We assume

$$(A1) \qquad M \neq \emptyset$$

Let $\underline{f}_k$, $k \in K$, be the global minimum of $f_k(x)$ subject to M or, in the nonconvex case, a sufficiently good lower bound. We assume that $\underline{f}_k$, $k \in K$, to be known. In the following we describe a well-known dialogue procedure (cf. e.g. [9], [12]) which constitutes the basis of our investigation. Let x^i be the currently computed feasible point. Then we consider a telescreen picture as described in Table 1.1.
The main information consists in estimating the objective values at the point x^i in comparison to $\underline{f}_k = \inf\{f_k(x) \mid x \in M\}, k \in K$.

$\underline{f}_1$	$f_1(x^i)$	$\dfrac{f_1(x^i) - \underline{f}_1}{\lvert\underline{f}_1\rvert}$	$\times$	100
$\vdots$		$\vdots$		
$\underline{f}_k$	$f_k(x^i)$	$\dfrac{f_k(x^i) - \underline{f}_k}{\lvert\underline{f}_k\rvert}$	$\times$	100
$\vdots$		$\vdots$		
$\underline{f}_l$	$f_l(x^i)$	$\dfrac{f_l(x^i) - \underline{f}_l}{\lvert\underline{f}_l\rvert}$	$\times$	100

Table 1.1

The third column contains the percentage of deviations of the current values $f_k(x^i)$ from the lower bounds $\underline{f}_k$. Clearly, we have to assign suitable values to these quantities in case of $\underline{f}_k = 0$ and $\underline{f}_i = -\infty$. The decision maker should answer the following questions by means of the telescreen pictures:

(i) Which f_k $(k \in K)$ do you wish to improve? Let $K_1 \subseteq K$ be the corresponding index set.

(ii) Which goals μ_k^1 do you wish for $f_k, k \in K_1$?

(iii) Which upper bound μ_k^1 do you accept for $f_k, k \in K \setminus K_1$?

μ^1 is the goal (the wishes of the decision maker), where $\mu_k^1, k \in K_1$, represent the so-called aspiration level and $\mu_k^1, k \in K \setminus K_1$, the reservation level.
We consider

$$M(\mu^1) := \{x \in M \mid f_k(x) \leq \mu_k^1, k \in K\},$$

and a point $\hat{x} \in M(\mu^1)$ is called a goal realizer. We will discuss the following questions:

Question 1: How useful are pathfollowing methods with the known starting point $x^0 \in M$ for finding a point $\hat{x} \in M(\mu^1)$ in case $M(\mu^1) \neq \phi$?

Question 2: How to obtain information that $M(\mu^1) = \phi$?
Of course, the two questions depend on each other. They were discussed for instance in [8], [9], [10], [7]. Here we consider various one-parametric optimization problems that are motivated by (locally) efficient points, (locally) efficient points with boundary ε, and (locally) weakly efficient points for the problem (MOP). We denote these sets by M_{eff} (M_{loceff}), M_{eff}^{ε} (M_{loceff}^{ε}), and M_{weff} ($M_{locweff}$), respectively.
First, we consider the following multiparametric optimization problem

$$P_1(\mu) : \min \left\{ \sum_{k \in K} \lambda_k^0 f_k(x) \mid x \in M, \ f_k(x) \leq \mu_k, \ k \in K \right\}, \mu \in I\!R^l,$$

where $\lambda^0 \in \Lambda := \{\lambda \in I\!R^l \mid \lambda_k > 0, k \in K\}$ is arbitrarily fixed. By $\psi_{glob}(\mu)(\psi_{loc}(\mu))$ we denote the set of all global (local) minimizers for $P_1(\mu)$. Then the following relation is known

$$M_{eff} = \bigcup_{\mu \in I\!R^l} \psi_{glob}(\mu) \quad (M_{loceff} = \bigcup_{\mu \in I\!R^l} \psi_{loc}(\mu))$$

(see e.g. [1], [3], [9]).
Then we obtain a one-parametric optimization problem by choosing a starting parameter μ^0 with

$$\mu_k^0 \geq f_k(x^i), \ k \in K,$$

and considering the connecting line

$$\left\{\mu \in I\!R^l \mid \mu = \mu^0 + t(\mu^1 - \mu^0), t \in [0,1]\right\}.$$

Then we obtain

$$P_1(t) := P_1(t, \mu^0, \mu^1) : \min \left\{ \sum_{k \in K} \lambda_k^0 f_k(x) \mid x \in M_1(t) \right\}, t \in [0,1],$$

where

$$M_1(t) := M_1(t, \mu^0, \mu^1) := \left\{x \in M \mid f_k(x) \leq \mu_k^0 + t(\mu_k^1 - \mu_k^0), k \in K\right\}.$$

Of course, using pathfollowing methods starting with x^0 and attaining $t = 1$ at a point $\hat{x}$ provides $\hat{x} \in M(\mu^1)$ because of $M_1(1) = M(\mu^1)$ (here we assume that $M(\mu^1) \neq \phi$). Unfortunately, we obtain such a point only for convex (MOP) with certainty. We have the same situation for some other parametrizations (see the references above). Regarding $P_1(t)$ in the non-convex case, $M_1(t)$ could be empty for all $t \in (t_1, t_2)$ and $(t_1, t_2) \subset [0, 1]$ (see Example 4.1 in Chapter 4). Then the decision maker does not know whether his goal μ^1 is a realistic one or not. In Example 4.1, $M(\mu^1)$ is not empty. From this point of view we propose a completely different parametrization, which is not motivated by the solution sets M_{eff} etc., but which has the advantage that the parameter-depending feasible set is non-empty and compact for all $t \in [0, 1]$ under the assumption that

$$(A2) \qquad M(\mu^1) \cap E(p) \neq \emptyset \quad \text{and} \quad x^0 \in I\!R^n \ \text{ arbitrarily chosen with } \ \|x^0\|^2 < p$$

where

$$E(p) := \{x \in R^n \mid \|x\|^2 \leq p\}$$

$$p > 0 \quad \text{sufficiently large.}$$

Now we consider the optimization problem

$$(P) \qquad \min\{\|x - x^0\|^2 \mid x \in M(\mu^1) \cap E(p)\}$$

and the following modified standard embedding

$$\begin{aligned} P_2(t) := P_2(t, x^0, \mu^1) : \min\{\|x - x^0\|^2 \mid \ & x \in M_2(t)\},\ t \in [0, 1] \\ M_2(t) := \{x \in I\!R^n \mid \ & t g_j(x) + (t-1) g_j^0 \leq 0,\ j \in J, \\ & t \tilde{f}_k(x) + (t-1) f_k^0 \leq 0,\ k \in K, \\ & \|x\|^2 - p \leq 0\},\ t \in [0, 1] \end{aligned}$$

where

$$\tilde{f}_k(x) := \tilde{f}_k(x, \mu_k^1) := f_k(x) - \mu_k^1,\ k \in K,$$

p sufficiently large, $f_k^0 > 0, k \in K$ and $g_j^0 > 0, j \in J$ with $f_{k_1}^0 \neq f_{k_2}^0, k_1, k_2 \in K, k_1 \neq k_2$ and $g_{j_1}^0 \neq g_{j_2}^0, j_1, j_2 \in J, j_1 \neq j_2$.

2 Theoretical Background and the Program Package PAFO.

First, we present a very short version of 2.5, 2.6 in [12]. We consider the general one-parametric problem:

$$P(t) \qquad \min\{f(x, t) | x \in M(t)\},\ t \in I\!R, \tag{2.1}$$

where $M(t) = \{x \in I\!R^n \mid h_i(x,t) = 0,\ i \in I,\ g_j(x,t) \leq 0,\ j \in J\}$, and $f, h_i, g_j \in C^q(I\!R^n \times I\!R, I\!R),\ i \in I, j \in J, q \geq 2$.
Furthermore, we introduce the following notations:

$$\begin{aligned}
\Sigma_{gc} &:= \{(x,t) \in I\!R^n \times I\!R \mid x \text{ is a generalized critical point}^1 \text{ (g.c. point) of } P(t)\},\\
\Sigma_{\text{stat}} &:= \{(x,t) \in I\!R^n \times I\!R \mid x \text{ is a stationary point of } P(t)\},\\
\Sigma_{\text{loc}} &:= \{(x,t) \in I\!R^n \times I\!R \mid x \text{ is a local minimizer of } P(t)\},\\
H &:= (h_1, \ldots, h_m)^T, \quad G := (g_1, \ldots, g_s)^T.
\end{aligned}$$

The Linear Independence Constraint Qualification (briefly LICQ) is satisfied at $\bar{x} \in M(\bar{t})$ if the vectors $D_x h_i(\bar{x}, \bar{t}),\ i \in I,\ D_x g_j(\bar{x}, \bar{t}),\ j \in J_0(\bar{x}, \bar{t})$, are linearly independent ($J_0(x,t) := \{j \in J \mid g_j(x,t) = 0\}$).
The Mangasarian-Fromovitz Constraint Qualification (briefly MFCQ) is satisfied at $\bar{x} \in M(\bar{t})$ if:

(MF1) $D_x h_i(\bar{x}, \bar{t}),\ i \in I$, are linearly independent,

(MF2) there exists a vector $\xi \in R^n$ with

$$\begin{aligned}
D_x h_i(\bar{x}, \bar{t})\xi &= 0, \qquad i \in I, \quad {}^2\\
D_x g_j(\bar{x}, \bar{t})\xi &< 0, \qquad j \in J_0(\bar{x}, \bar{t}).
\end{aligned}$$

The KKT-system for a given problem $P(t)$ is fulfilled at a point $(\bar{x}, \bar{t})$ if there exists a point $\bar{y} \in I\!R^{m+s}$ such that $\mathcal{H}(\bar{x}, \bar{y}, \bar{t}) = 0$, where $\mathcal{H} : I\!R^{n+m+s+1} \to I\!R^{n+m+s}$ is defined by

$$\mathcal{H}(x,y,t) = \left\{ \begin{array}{l} D_x f(x,t) + \sum\limits_{i \in I} y_i D_x h_i(x,t) + \sum\limits_{j \in J} y_{m+j}^+ D_x g_j(x,t) \\ -h_i(x,t), i \in I \\ y_{m+j}^- - g_j(x,t), j \in J \end{array} \right\} \tag{2.2}$$

(for $\alpha \in I\!R$ let $\alpha^+ = \max\{\alpha, 0\}$ and $\alpha^- = \min\{\alpha, 0\}$). Obviously, the so-called Kojima-mapping $\mathcal{H}$ in (2.2) is piecewise continuously differentiable. In [17] the classical definition of a regular value of a continuously differentiable function is generalized for piecewise continuously differentiable functions. Furthermore, it is shown that if $0 \in I\!R^{n+m+s}$ is a regular value of $\mathcal{H}$, then the set $\mathcal{H}^{-1}(0)$ is a piecewise one-dimensional C^1-manifold (briefly PC^1-manifold).

Next, we cite our short characterization from 2.5 in [12] of the class $\mathcal{F}$ introduced by Jongen, Jonker and Twilt ([15, 16]). In [16] the local structure of Σ_{gc} is completely

[1] For the definition we refer to [16], see [12], too
[2] We consider the gradient $D_x h_i(\bar{x}, \bar{t})$ as a row vector.

described if (f, H, G) belongs to a C^3_s-open and dense subset $\mathcal{F}$ of $C^3(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})^{1+m+s}$, where C^3_s denotes the strong (or Whitney-) C^3-topology (see [12], too).
If $(f, H, G) \in \mathcal{F}$, then $\Sigma_{\rm gc}$ can be divided into 5 types.

Type 1: A point $\bar z = (\bar x, \bar t) \in \Sigma_{\rm gc}$ is of Type 1 if the following conditions are satisfied: There exist $\bar\lambda_i, \bar\mu_j \in \mathbb{R}$, $i \in I$, $j \in J_0(\bar z)$ with

$$\Big(D_x f + \sum_{i\in I} \bar\lambda_i D_x h_i + \sum_{j\in J_0(\bar z)} \bar\mu_j D_x g_j\Big)\big|_{z=\bar z} = 0, \tag{2.3}$$

LICQ is satisfied at $\bar x \in \mathbf{M}(\bar t)$, (2.4a)

(therefore $\bar\lambda_i$, $\bar\mu_j$, $i \in I$, $j \in J_0(\bar z)$ are uniquely defined)

$\bar\mu_j \neq 0, \quad j \in J_0(\bar z)$, (2.4b)

$D^2_x L(\bar x, \bar t)|_{T(\bar z)}$ is nonsingular, (2.4c)

where $D^2_x L$ is the Hessian of the Lagrangian

$$L(x,t) = f(x,t) + \sum_{i\in I} \bar\lambda_i h_i(x,t) + \sum_{j\in J_0(\bar z)} \bar\mu_j g_j(x,t),$$

and the uniquely determined numbers $\bar\lambda_i, \bar\mu_j$ are taken from (2.3). Furthermore,

$$T(z) = \{\xi \in \mathbb{R}^n \mid D_x h_i(z)\xi = 0, i \in I, D_x g_j(z)\xi = 0, j \in J_0(z)\}$$

is the tangent space at z. $D^2_x L(z)|_{T(z)}$ represents $V^T D^2_x L V$, where V is a matrix whose columns form a basis of $T(z)$.
A point of Type 1 is a nondegenerate critical point. The set $\Sigma_{\rm gc}$ is the closure of the set of all points of Type 1, the points of the Types 2–5 constitute a discrete subset of $\Sigma_{\rm gc}$. The points of the Types 2–5 represent basic degeneracies:

Type 2 – violation of (2.4b)

Type 3 – violation of (2.4c)

Type 4 – violation of (2.4a) and $|I| + |J_0(\bar z)| - 1 < n$

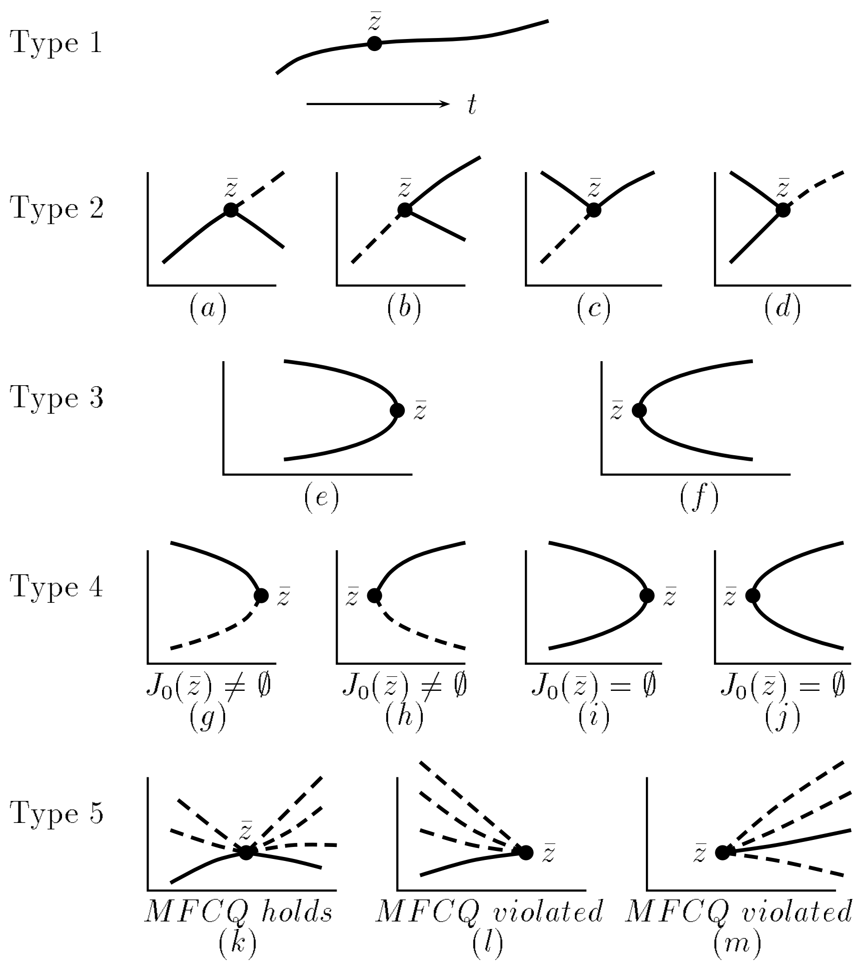
Type 5 – violation of (2.4a) and $|I| + |J_0(\bar z)| = n + 1$.

The full curve stands for the curve of stationary points $z = (x, t)$, and the dotted curve represents the curve of g.c. points which are not stationary points.

For each of these five types Figure 2.1 illustrates the local structure of $\Sigma_{\rm gc}$ in the neighbourhood of stationary points. Let $\Sigma^\nu_{\rm gc}$, $\nu \in \{1, \ldots, 5\}$ be the set of g.c. points of Type ν. Figure 2.2 illustrates the local structure of $\mathcal{F}$ in $\Sigma_{\rm loc}$ and $\Sigma_{\rm stat}$. The class $\mathcal{F}$ is defined by

$$\mathcal{F} = \Big\{(f, H, G) \in C^3(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})^{1+m+s} \mid \Sigma_{\rm gc} \subset \bigcup_{\nu=1}^{5} \Sigma^\nu_{\rm gc}\Big\}.$$

The full curve stands for a curve of local minimizers and the dotted curve in Fig. 2.2(c), (d), (e), (f) represents a curve of stationary points not being local minimizers. The dotted curve in Fig. 2.2(g), (h) stands for a curve of stationary points in case $J_0(\bar x, \bar t) = \emptyset$.



Type 1
$\bar{z}$
t
Type 2
(a)
(b)
(c)
(d)
Type 3
(e)
(f)
Type 4
$J_0(\bar{z}) \neq \emptyset$
(g)
$J_0(\bar{z}) \neq \emptyset$
(h)
$J_0(\bar{z}) = \emptyset$
(i)
$J_0(\bar{z}) = \emptyset$
(j)
Type 5
$MFCQ$ $holds$
(k)
$MFCQ$ $violated$
(l)
$MFCQ$ $violated$
(m)

Figure 2.1

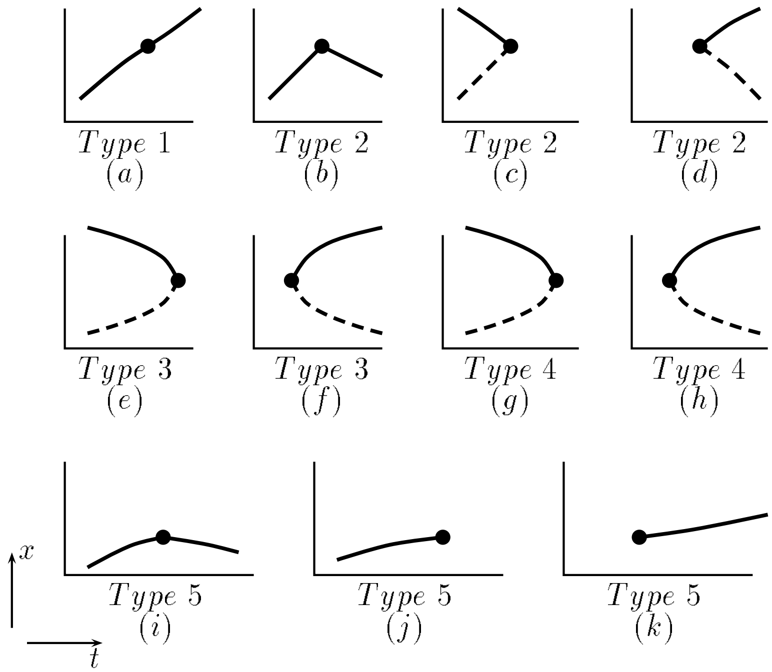


Figure 2.2

The following theorem provides a special perturbation of (f, H, G) with additional parameters that can be chosen arbitrarily small such that the perturbed function vector belongs to the class $\mathcal{F}$.

Theorem 2.1 (cf. [18]). *Let $(f, H, G) \in C^3(\mathbb{R}^n \times R, R^{1+m+s})$. Then, for almost all $(b, A, c, D, e, F) \in \mathbb{R}^n \times \mathbb{R}^{n(n+1)/2} \times \mathbb{R}^m \times \mathbb{R}^{mn} \times \mathbb{R}^s \times \mathbb{R}^{sn}$, we have*

$$(f(x,t) + b^T x + x^T A x, H(x,t) + c + Dx, G(x,t) + e + Fx) \in \mathcal{F}.$$

Here "almost all" means: each measurable subset of

$$\{(b, A, c, D, e, F) \mid (f(x,t) + b^T x + x^T A x, H(x,t) + c + Dx, G(x,t) + e + Fx) \notin \mathcal{F}\}$$

has the Lebesgue-measure zero. □

Remark 2.2 (cf. [18]). Considering Σ_{stat} we note that the condition $(f, H, G) \in \mathcal{F}$ implies that zero is a regular value of the Kojima-mapping $\mathcal{H}$. □

Definition 2.3 Let $K \subseteq \mathbb{R} \cup \{\pm\infty\}$.

(i) The problem $P(t)$ is called regular in the sense of Jongen-Jonker-Twilt – briefly JJT-regular – (with respect to K) if $(f, H, G) \in \mathcal{F}\Big((\mathbb{R}^n \times K) \cap \Sigma_{\text{gc}} \subset \bigcup_{\nu=1}^{5} \Sigma_{\text{gc}}^{\nu}\Big)$.

(ii) The problem $P(t)$ is called regular in the sense of Kojima-Hirabayashi – briefly KH-regular – (with respect to K) if $0 \in \mathbb{R}^{n+m+s}$ is a regular value of $\mathcal{H}$ ($\mathcal{H}|_{\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^s \times K}$).

Theorem 2.4 (cf. [17]). *Let* $(f, H, G) \in C^3(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})^{1+m+s}$. *Then, for almost all* $(b, c, d) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^s$, *the problem*

$$P_{(b,c,d)}(t) \quad \min\left\{ f(x,t) + b^T x \;\middle|\; \begin{array}{ll} h_i(x,t) + c_i = 0, & i \in I, \\ g_j(x,t) + d_j \leq 0, & j \in J \end{array} \right\}$$

is KH-regular. □

Now, we present two theorems that are essential for our analysis.

Theorem 2.5 (cf. [4]). *We assume*

(C1) *$M(t)$ is non-empty and there exists a compact set C with $M(t) \subseteq C$ for all $t \in [0, 1]$.*

(C2) *$P(t)$ is KH-regular with respect to $[0, 1]$.*

(C3) *There exists a $t_1 > 0$ and a continuous function $x : [0, t_1) \to \mathbb{R}^n$ such that $x(t)$ is the unique stationary point for $P(t)$ for $t \in [0, t_1)$.*

(C4) *MFCQ is satisfied for all $x \in M(t)$ for all $t \in [0, 1]$.*

Then there exists a PC^1-path in Σ_{stat} that connects $(x^0, 0)$ with some point $(x^*, 1)$. □

Applying Remark 2.2 we obtain

Corollary 2.6 *We assume* (C1), (C3), (C4) *and*

(D2) *$P(t)$ is JJT-regular with respect to $[0, 1]$.*

Then there exists a PC^2-path $K(x^0, 0)$ in Σ_{stat} connecting $(x^0, 0)$ with some point $(x^*, 1)$. Furthermore if $(x, t) \in K(x^0, 0)$ then (x, t) belongs to $\bigcup_{\nu \in \{1,2,3,5\}} \Sigma^\nu_{\text{gc}}$. □

Remark 2.7 Assume (C4). Now we have a look at Fig. 2.2. Since the MFCQ is satisfied, points of Type 5 in (j) and (k) are excluded. □

Finally we present a consequence of a general topological stability result given in [13]:

Theorem 2.8 (cf. [13]). *We assume* (C1) *and* (C4). *Then $M(t_1)$ is homeomorphic with $M(t_2)$ for all $t_1, t_2 \in [0, 1]$.* □

On the program package PAFO (this is a very short version of 4.5 and 5.2 in [12])

PAFO (cf. [19] and [5]) is based on a pathfollowing method (called PATH III in 4.5 [12]) and jumps (called JUMP I in 5.2 [12] and JUMP II in 5.3 [12]).
We explain the main ideas of PATH III and JUMP I, but not those of JUMP II as we do not need them here.

PATH III

This algorithm computes a numerical description of a compact connected component in $\sum_{gc}$, i.e., in particular it finds a discretization of an interval $[t_A, t_B]$, $t_A < 0 < t_B$ (not necessarily $[t_A, t_B] \supset [0, 1]$), and corresponding g.c. points starting at $(x^0, 0) \in \sum_{gc}$ (cf. (A2)). The algorithm is based on the active index set strategy and is a so-called predictor-corrector scheme if the active index set is constant. A Newton corrector is used.
The main point of the approach consists in the computation of the new index sets for the possible continuations.
We note that we do not have any numerical difficulties walking around turning points of the Types 3 or 4.
More precisely: If there exists a PC^2-path connecting $(x^0, 0)$ and a point $(x^*, 1)$, then we obtain, in a finite number of predictor and corrector steps, a point lying in the radius of convergence of the Newton method for x^* with respect to the problem P(1).
Since PATH III is not successful in finding a point $(x^*, 1) \in \sum_{gc}$ in general, we propose to jump from one connected component in $cl \sum_{loc}$ and $\sum_{gc}$, respectively, to another one.

JUMP I

This algorithm works in the set $cl\ \sum_{loc}$. Starting at the known local minimizer x^0 at $t_0 = 0$, a connected component in $cl\ \sum_{loc}$ for increasing t will be numerically described by using PATH III. Depending on the appearance of a singularity, a direction of descent will be computed. Using a feasible direction method, a local minimizer on another connected component in $\sum_{loc}$ will be calculated and PATH III starts again. We have to take into account that we have no proposals for jumps in any case. Jumps are possible if a turning point of Type 2 or a point of Type 3 occur. Let t be near $\bar{t}$, $t < \bar{t}$, and let $x_m(t)$ and $x_s(t)$ be the local minimizer and a point of $\sum_{stat} \setminus \sum_{loc}$, respectively. Then, as t tends to $\bar{t}$, the vector

$$u(t) := \frac{x_s(t) - x_m(t)}{\|x_s(t) - x_m(t)\|}$$

tends to a tangential vector, say $\bar{u}$, which is a direction of (higher order) descent (cf. Fig. 2.3 for a point of Type 3).
Hence, for t near $\bar{t}, t < \bar{t}$, the vector $x_s(t) - x_m(t)$ provides an approximately tangential direction of descent (cf. Fig. 2.3).
A g.c. point of Type 4 is a quadratic turning point and, when passing $\bar{z}$ along $cl\ \sum_{loc}$, the local minimizer switches into a local maximizer. We have the following cases for

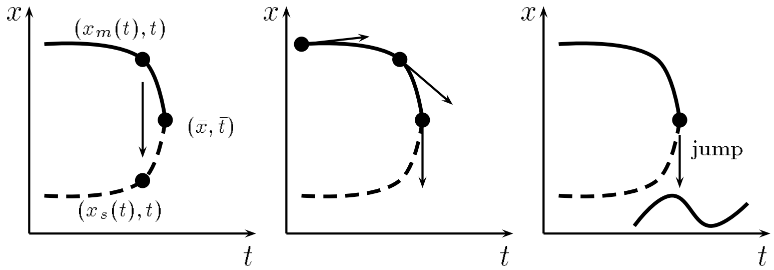


Figure 2.3

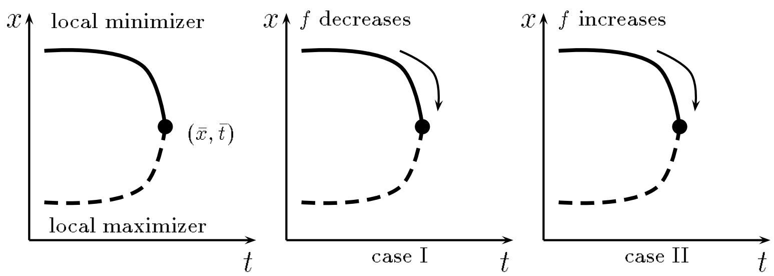


Figure 2.4

$t < \bar{t}$ and t close to $\bar{t}$:

Case I : The value of f decreases
Case II: The value of f increases

In Case I it is possible to jump to another branch of local minimizers. In fact, since the feasible set $M(t)$ is compact, we compute a point on $\sum_{gc}$ beyond the turning point, say $(x_{max}(t), t)$ with $t < \bar{t}, t$ close to $\bar{t}$. The point $x_{max}(t)$ is a local maximizer for $P(t)$ and we can start at $x_{max}(t)$ with a descent method in order to find a local mimimizer. In Case II, there is no proposal for possible jumps (cf. Fig. 2.4).
If a point of Type 5 appears, we also do not know a jump in case the MFCQ is violated. Such a situation is characterized by the fact that the connected component in the feasible set shrinks to one point and becomes empty for increasing t.

3 Properties of the standard embedding

The first theorem includes basic properties of the standard embedding $P_2(t)$. We have

Theorem 3.1 *Assume (A1) and (A2). Then $P_2(t)$ has the following properties*

(i) $M_2(t)$ is nonempty and compact for all $t \in [0, 1]$

(ii) $M_2(1) = M(\mu^1) \cap E(p)$

(iii) x^0 *is a global minimizer, the only stationary point and non-degenerated for* $P_2(0)$.

If we assume, moreover, that

$(A3)$ $\quad P_2(t)$ *is JJT-regular with respect to* $(0,1]$,

then all singularities may appear.

Remark 3.2 $P_2(t)$ is constructed in such a way that the starting point has nice properties. We cannot come back to $t = 0$ in $\sum_{stat}$, but the following difficulties may arise, where we do not attain $t = 1$.

a) A point of Type 4 (Case II) appears and we do not have a jump in $\sum_{stat}$ to another connected component (cf. Example 4.2)

b) A point of Type 5 (where the MFCQ is not satisfied) appears and we do not have a jump.

Under the following additional assumption:

(A4) $\quad$ MFCQ is satisfied for all $x \in M_2(t)$ for all $t \in [0,1]$

the difficulties mentioned above are excluded.

Theorem 3.3 *Assume (A1) - (A4). Then there exists a* PC^2*-path in* $\sum_{stat}$ *connecting* $(x^0, 0)$ *and* $(x^*, 1)$ *with* $x^* \in M(\mu^1)$ *and points of the Types 1,2,3, and 5 (MFCQ is satisfied) may appear.* □

Proof:
We check the assumptions of Corollary 2.6 □

Remark 3.4 In a finite number of predictor and corrector steps PAFO computes a point lying in the radius of Newton methods with respect to x^*, i.e. we have at least a superlinear rate of convergence.

If we replace (A3) by the weaker assumption

(A3)' $\quad P_2(t)$ is KH-regular with respect to $(0,1]$,

then we can use Theorem 2.5 and we obtain

Theorem 3.5 *Assume (A1), (A2), (A3)', and (A4). Then there exists a* PC^1*-path in* $\sum_{stat}$ *connecting* $(x^0, 0)$ *and* $(x^*, 1)$ *with* $x^* \in M(\mu^1)$. □

The Examples 4.1 and 4.2. illustrate Theorem 3.3.

In the following we discuss the assumptions (A3) and (A3)' and ask how large the classes have to be for (A3) resp. (A3)' to be satisfied. Let $F = (f_1, \ldots, f_l)^T$ and $G = (g_1, \ldots, g_s)^T$. Then we consider the mapping $\phi : C^3(I\!R^n, I\!R)^{s+l+2} \to C^3(I\!R^n \times I\!R, I\!R)^{s+l+2}$ defined by

$$\phi(\|x-x^0\|^2, G, F, \|x\|^2 - p) = \begin{bmatrix} \|x-x^0\|^2 \\ tg_1(x) + (t-1)g_1^0 \\ \vdots \\ tg_s(x) + (t-1)g_s^0 \\ t\tilde{f}_1(x) + (t-1)f_1^0 \\ \vdots \\ t\tilde{f}_l(x) + (t-1)f_l^0 \\ \|x\|^2 - p \end{bmatrix}$$

by using the embedding $P_2(t)$ for the problem (P). In addition to the class $\mathfrak{F}|_K$ (JJT-regular with respect to K), we introduce the class $\mathfrak{K}|_K$ (KH-regular with respect to K).

Theorem 3.6 *(i) Let $(F,G) \in C^3(I\!R^n, I\!R)^{s+l}$. Then $\phi^{-1}(\mathfrak{F}|_{[0,1]})$ is C^3_s-open and $\phi^{-1}(\mathfrak{F}|_{(0,1]})$ is C^3_s-dense in $C^3(I\!R^n, I\!R)^{s+l+2}$*

(ii) Let $(F,G) \in C^2(I\!R^n, I\!R)^{s+l}$. Then $\phi^{-1}(\mathfrak{K}|_{[0,1]})$ is C^2_s-open and $\phi^{-1}(\mathfrak{K}|_{(0,1]})$ is C^2_s-dense in $C^2(I\!R^n, I\!R)^{s+l+2}$. □

For the proof of Theorem 3.6 we study Theorem 2.1 specified by the embedding $P_2(t)$, i.e., we define

$$\left.\begin{array}{lcl} f(x,t) & := & \|x-x^0\|^2 \\ g_j(x,t) & := & tg_j(x) + (t-1)g_j^0, \ j = 1, \ldots, s \\ g_{s+k}(x,t) & := & t\tilde{f}_k(x) + (t-1)f_k^0, \ k = 1, \ldots, l \\ g_{s+l+1}(x,t) & := & \|x\|^2 - p \end{array}\right\} \quad (3.1)$$

Let A be an $n \times n$ symmetric matrix ($A \in I\!R^{\frac{1}{2}n(n+1)}$), $b^j \in I\!R^n$, $j = 1, \ldots, s+l+1$, $c \in I\!R^n$, $d_j \in I\!R$, $j = 1, \ldots, s+l+1$

$$d := (d_1, \ldots, d_{s+l})^T \in I\!R^{s+l}, \ \mathfrak{B} := (b^1, \ldots, b^{s+l+1}, d),$$

$$\mathfrak{D} := (A, C, \mathfrak{B}) \in \mathbb{R}^{\gamma}, \gamma := \frac{1}{2}n(n+1) + n(s+l+1) + s + l$$

and we define

$$f(x,t,A,c) := f(x,t) + x^T A x + c^T x,$$

$$g_j(x,t,b^j,d_j) := g_j(x,t) + t b^{j^T} x + t d_j, \ j = 1, \ldots, s+l.$$

$$g_{s+l+1}(x,t,b^{s+l+1},d_{s+l+1}) := \|x\|^2 - p + (b^{s+l+1})^T x + d_{s+l+1}$$

We consider the problem

$$P_{\mathfrak{D}}(t) : \min\{f(x,t,A,c) \mid x \in M(\mathfrak{B},t)\}, \ t \in \mathbb{R},$$

where

$$M(\mathfrak{B},t) = \{x \in \mathbb{R}^n \mid g_j(x,t,b^j,d_j) \leq 0, \ j = 1, \ldots, s+l+1\}.$$

Proposition 3.7 *Let $(F,G) \in C^3(\mathbb{R}^n, \mathbb{R})^{s+l}$. Then $P_{\mathfrak{D}}(t)$ is JJT-regular for almost all $\mathfrak{D} \in \mathbb{R}^{\gamma}$ with respect to $(0,1]$.* □

The proof follows the same lines as the proof of Theorem 2.1 in [18]. □

Using the special functions in (3.1) once more, we obtain

Proposition 3.8 *Let $(F,G) \in C^2(\mathbb{R}^n, \mathbb{R})^{s+l}$. Then, for almost all $(b,d) \in \mathbb{R}^n \times \mathbb{R}^{s+l}$, the problem $P_{(b,d)}(t) : \min\{f(x,t) + b^T x \mid g_j(x,t) + t d_j \leq 0, j = 1, \ldots, s+l, g_{s+l+1}(x,t) + d_{s+l+1} \leq 0\}$ is KH-regular with respect to (0,1].* □

For the proof see Theorem 2.4 and the hints for the proof in Chapter 8 in [17].

Proof of Theorem 3.6:
(i) a) $\phi^{-1}(\mathfrak{F}|_{(0,1]})$ is C^3_s-dense in $X := C^3(\mathbb{R}^n, \mathbb{R})^{s+l+2}$. Let $H := (\|x - x^0\|^2, G, F, \|x\|^2 - p) \in X$. We have to show that for any ε-neighbourhood $V^3_{\varepsilon,H}$ of H in the C^3_s (or strong C^3) topology (cf. e.g. Chapter 2 in [12]) there exists an $H' \in V^3_{\varepsilon,H} \cap \phi^{-1}(\mathfrak{F}|_{(0,1]})$, in other words,

$$H' \in V^3_{\varepsilon,H} \quad \text{and} \quad \phi(H') \in \mathfrak{F}|_{(0,1]}. \tag{3.2}$$

Using the theorem on the partitioning of unity, we denote $B_\varrho := \{x \in \mathbb{R}^n \mid \|x\| < \varrho\}$ and consider the open covering of $\mathbb{R}^n$ given by $\{B_3, \mathbb{R}^n \setminus cl\ B_2)$ and a corresponding partition of unity $\{\xi^1_1, \xi^2_1\}$,

$$\xi^i_1 : \mathbb{R}^n \longrightarrow \{0,1\}, \quad i = 1,2.$$

That means

- $supp\ \xi_1^1 \subseteq B_3$, $supp\ \xi_1^2 \subseteq I\!R^n \setminus cl\ B_2$, where $supp\ \xi_1^i = cl\{x \in I\!R^n \mid \xi_1^i(x) > 0\}$, $i = 1, 2$.

- $\xi_1^1(x) + \xi_1^2(x) = 1$ for all $x \in I\!R^n$.

Then $\xi^1_{1\ |\ cl\ B_2} \equiv 1$ and $\xi^2_{1\ |\ I\!R^n \setminus B_3} \equiv 0$. This partition of unity exists independent of H and $V^3_{\varepsilon,H}$. Now, we construct H^1:

$$H^1(x) := \begin{bmatrix} h_0^1(x) \\ h_1^1(x) \\ \vdots \\ h_s^1(x) \\ h_{s+1}^1(x) \\ \vdots \\ h_{s+l}^1(x) \\ h_{s+l+1}^1 \end{bmatrix}$$

where

$$\begin{array}{lcl} h_0^1(x) & := & \xi_1^1(x)\left[\|x - x^0\|^2 + x^T A^1 x + c^{1T} x\right] + \xi_1^2(x)\|x - x^0\|^2 \\ h_j^1(x) & := & \xi_1^1(x)\left[g_j(x) + (b^j)^{1T} x + d_j^1\right] + \xi_1^2(x) g_j(x),\ j = 1, \ldots, s, \\ h_{s+k}^1(x) & := & \xi_1^1(x)\left[\tilde f_k(x) + (b^{s+k})^{1T} x + d_{s+k}^1\right] + \xi_1^2(x)\tilde f_k(x),\ k = 1, \ldots, l, \\ h_{s+l+1}^1(x) & := & \xi_1^1(x)\left[\|x\|^2 - p + \left(b^{s+l+1}\right)^{1T} x + d_{s+l+1}^1\right] + \xi_1^2(\|x\|^2 - p). \end{array}$$

Now, by Propsition 3.7 we can choose $\mathfrak{D} \in I\!R^\gamma$ sufficiently small to have

1. $H^1 \in V^3_{\varepsilon,H}$,

2. $\sum_{gc}(\phi(H^1)) \cap (cl\ B_2 \times (0,1]) \subseteq \bigcup\limits_{\nu=1}^{5} \sum_{gc}^{\nu} \phi(H^1))$.

Now consider the open covering of $I\!R^n$ given by $\{B_4 \setminus cl\ B_1, B_2 \cup \{I\!R^n \setminus\ cl\ B_3\}\}$ and a corresponding partition of unity $\{\xi_2^1, \xi_2^2\}$, which exists independent of H, H^1 and $V^3_{\varepsilon,H}$. Taking into account the first part and Proposition 3.7, we can choose another $\mathfrak{D}^2 \in I\!R^\gamma$ sufficiently small to obtain

1. $H^2 \in V^3_{\varepsilon,H}$, where

$$H^2(x) := \begin{bmatrix} h_0^2(x) \\ h_1^2(x) \\ \vdots \\ h_s^2(x) \\ h_{s+1}^2(x) \\ \vdots \\ h_{s+l}^2(x) \\ h_{s+l+1}^2 \end{bmatrix}$$

$$\begin{aligned} h_0^2(x) &:= \xi_2^1(x)\left[h_0^1(x) + x^T A^2 x + c^{2T}x\right] + \xi_2^2(x)h_0^1(x), \\ h_j^2(x) &:= \xi_2^1(x)\left[h_j^1(x) + (b^j)^{2T}x + d_j^2\right] + \xi_2^2(x)h_j^1(x),\ j = 1,\dots,s, \\ h_{s+k}^1(x) &:= \xi_2^1(x)\left[h_k^1(x) + (b^{s+k})^{2T}x + d_{s+k}^2\right] + \xi_2^2(x)h_k^1(x),\ k = 1,\dots,l+1. \end{aligned}$$

2. $\sum_{gc}(\phi(H^2)) \cap (cl\ B_3 \times (0,1]) \subseteq \bigcup\limits_{\nu=1}^{5} \sum_{gc}^{\nu}(\phi(H^2))$,
3. $H^2 \mid_{cl\ B_1} \equiv H^1 \mid_{cl\ B_1}$.

In this way, we obtain a sequence of functions H^ϱ such that

1. $H^\varrho \in V^3_{\varepsilon,H}$
2. $\sum_{gc}(\phi(H^\varrho)) \cap (cl\ B_{\varrho+1} \times (0,1]) \subseteq \bigcup\limits_{\nu=1}^{5} \sum_{gc}^{\nu}(\phi(H^\varrho))$,
3. $H^\varrho \mid_{cl\ B_{\varrho-1}} \equiv H^{\varrho-1} \mid_{cl\ B_{\varrho-1}}$

For $x \in I\!R^n$ we define $\varrho(x) := \min\{\varrho \mid x \in B_\varrho\}$ and $H'(x) := H^{\varrho(x)}(x)$ and we obtain (3.2).
(i) b) $\phi^{-1}(\mathfrak{F}|_{[0,1]}$ is C^3_s-open in X.
Since $\mathfrak{F}|_{[0,1]}$ is C^3_s-open in $Y|_{[0,1]} := C^3(I\!R^n \times [0,1], I\!R)^{s+l+2}$, it is sufficient that $\phi : X \to Y|_{[0,1]}$ is continuous. We denote $\phi := (\phi_0, \phi_1, \dots, \phi_{s+l+1})$. It is sufficient to consider $\tilde{\phi} := (\phi_1, \dots, \phi_{s+l})$.
For $(G', F') \in X$ it holds

$$\left.\begin{aligned} |\phi_j(G,F) - \phi_j(G',F')| &= |t(g_j(x) - g_j'(x))|,\ j = 1,\dots,s \\ |\phi_{s+k}(G,F) - \phi_{s+k}(G',F')| &= |t(f_k(x) - f_k'(x))|,\ k = 1,\dots,l. \end{aligned}\right\} \qquad (3.3)$$

Let $(G,F) \in X$ be fixed and let $V^3_{\varrho,\tilde{\phi}(G,F)}$ be a ϱ-neighbourhood of $\tilde{\phi}(G,F)$ in $Y|_{[0,1]}$. Here $\varrho(x,t) = (\varrho_1(x,t),\ldots,\varrho_{s+l}(x,t))$, where $\varrho_\nu(x,t) \in C^0(\mathbb{R}^n \times [0,1], \mathbb{R}_+), \nu = 1,\ldots,s+l$.
We have to prove that there exists an ε-neighbourhood $V^3_{\varepsilon,(G,F)}$ of (G,F) in X such that

$$\tilde{\phi}(V^3_{\varepsilon,(G,F)}) \subseteq V^3_{\varrho,\tilde{\phi}(G,F)} \tag{3.4}$$

Here, $\varepsilon(x) := (\varepsilon_1(x),\ldots,\varepsilon_{s+l}(x))$, where $\varepsilon_\nu \in C^0(\mathbb{R}^n,\mathbb{R}_+), \nu = 1,\ldots,s+l$.
Put $\varepsilon_\nu(x) := \min_{0\le t\le 1} \varrho_\nu(x,t), \nu = 1,\ldots,s+l$. Then $\varepsilon_\nu \in C^0(\mathbb{R}^n,\mathbb{R}_+)$ and $\varepsilon_\nu(x) \le \varrho_\nu(x,t)$ for all $(x,t) \in \mathbb{R}^n \times [0,1], \nu = 1,\ldots,s+l$. Using (3.3), the inclusion (3.4) is fulfilled and, therefore, $\tilde{\phi}$ and also ϕ are continuous due to continuity arguments of the functions G and F.
(ii) a) $\phi^{-1}(\mathfrak{K}|_{[0,1]})$ is C^2_s-dense in $C^2(\mathbb{R}^n, \mathbb{R}\times\mathbb{R}^s\times\mathbb{R}^l\times\mathbb{R})$. We follow the same concept as in (i) a) using Proposition 3.8.
(ii) b) $\phi^{-1}(\mathfrak{K}|_{(0,1]})$ being C^2_s-open in $C^2(\mathbb{R}^n, \mathbb{R}\times\mathbb{R}^s\times\mathbb{R}^l\times\mathbb{R})$ follows by continuity arguments with respect to ϕ. □

We note that such a kind of theorem is proposed e.g. in [6] for another embedding with respect to justifying (A3).

Now we ask how we can enter the class $\mathfrak{F}$ and $\mathfrak{K}$, respectively. We consider

$$\begin{aligned} P_{\mathfrak{B}}(t) : \min\Big\{(x-x^0)^T A(x-x^0) \ \ | \quad & tg_j(x) + (t-1)(g^0_j + b^{j^T}x + d_j) \le 0,\ j \in J, \\ & t\tilde{f}_k(x) + (t-1)(f^0_k + \tilde{b}^{k^T}x + \tilde{d}_k) \le 0,\ k \in K, \\ & \|x\|^2 + c^Tx - p \le 0\Big\} \end{aligned}$$

$$\begin{aligned} \mathfrak{B} &= (A, x^0, B, d, \tilde{B}, \tilde{d}, c, p) \\ B &:= (b^1,\ldots,b^s),\ \tilde{B} := (\tilde{b}^1,\ldots,\tilde{b}^l),\ d = (d_1,\ldots,d_s)^T, \tilde{d} = (\tilde{d}_1,\ldots,\tilde{d}_l)^T, \end{aligned}$$

Let $\mathfrak{A} \subseteq \mathbb{R}^{\frac{1}{2}n(n+1)}$ be the set of all non-singular symmetric (n,n)-matrices. Then $\mathfrak{A}$ is open in $\mathbb{R}^{\frac{1}{2}n(n+1)}$ and $\mathbb{R}^{\frac{1}{2}n(n+1)} \setminus \mathfrak{A}$ has the Lebesgue measure 0.
We have (see the proof of Theorem 3.6)

Corollary 3.9 *Let $(F,G) \in C^3(\mathbb{R}^n,\mathbb{R})^{s+l}$. Then, $P_{\mathfrak{B}}(t)$ is JJT-regular with respect to (0,1) for almost all $(A,x^0,B,d,\tilde{B},\tilde{d},c,p) \in \mathfrak{A} \times \mathbb{R}^n \times \mathbb{R}^{n\cdot s} \times \mathbb{R}^s \times \mathbb{R}^{n\cdot l} \times \mathbb{R}^l \times \mathbb{R}$.*

Remark 3.10 For the starting situation $(t = 0)$ we have to choose A to be positive definite and $b^j, d_j, j \in J, \tilde{b}^k, \tilde{d}_k, k \in K$, in such a way that $b^{j^T}x^0 + d_j < 0, j \in J$, and $\tilde{b}^{k^T}x^0 + \tilde{d}_k < 0, k \in K$. Then x^0 is a global minimizer, the only stationary point, and non-degenerated.

Now we consider

$$P_{\mathfrak{C}}(t) : \min \{\|x - x^0\|^2 \mid \begin{array}{l} t g_j(x) + (t-1)(g_j^0 + d_j) \le 0,\ j \in J, \\ t\tilde{f}_k(x) + (t-1)(f_k^0 + \tilde{d}_k) \le 0,\ k \in K, \\ \|x\|^2 - p \le 0\}, \end{array}$$

where $\mathfrak{C} := (x^0, d, \tilde{d}, p)$.

Corollary 3.11 *Let $(F,G) \in C^2(I\!R^n, I\!R)^{s+l}$. Then, $P_{\mathfrak{C}}(t)$ is KH-regular with respect to (0,1) for almost all $\mathfrak{C} \in I\!R^{nsl+1}$ with $d_j > g_j^0, j \in J$ and $\tilde{d}_k > f_k^0, k \in K$ and $p > \|x^0\|^2$.*

Remark 3.12
(i) x^0 is a global minimizer, the only stationary point, and non-degenerated.
(ii) If we choose $p > \|x^0\|$ sufficiently large, the feasible set of $P_{\mathfrak{C}}(t)$ is non-empty and compact for all $t \in [0,1)$.

Finally, we discuss the assumption (A4). This is a condition to the parameter-depending feasible set $M_2(t)$ for all $t \in [0,1]$.
First, we ask for a sufficient condition with respect to the set $M(\mu^1) \cap E(p)$, which we will call, as in other papers (cf. e.g. [4] [11], [2]), the Enlarged Mangasarian-Fromovitz Constraint Qualification (briefly EnMFCQ).
Let $(F,G) \in C^1(I\!R^n, I\!R)^{s+l}$. The EnMFCQ for $M(\mu^1) \cap E(p)$:
For all $x \in E(p)$ it holds: There exists a $\xi \in I\!R^n$ with

(i) $g_j(x) + Dg_j(x)\,\xi < 0,\ j \in \{j \in J \mid g_j(x) \ge 0\}$

(ii) $\tilde{f}_k(x) + Df_k(x)\,\xi < 0,\ k \in \{k \in K \mid \tilde{f}_k(x) \ge 0\}$

(iii) $2x^T\xi < 0$ if $\|x\| = p$

Theorem 3.13 *Let $(F,G) \in C^1(I\!R, I\!R)^{s+l}$. Assume (A2) and the EnMFCQ. Then the MFCQ is satisfied for all $x \in M_2(t)$ for all $t \in [0,1]$.*

The proof runs along the lines of the proof of Theorem 10 in [2].

Remark 3.14 Using Theorem 2.8 we obtain that $M_2(0)$ is homeomorphic to $M_2(1) = M(\mu^1) \cap E(p)$ and $M_2(0)$ is a convex set. This shows how restrictive the assumption EnMFCQ is.

Second, we ask for a necessary and sufficient condition, where we follow the idea described for other embeddings in [11]. We know that the starting point x^0 for $P_2(0)$ (the only stationary point, cf. Theorem 3.1) lies on a uniquely determined connected component $C(x^0, 0)$ in $\sum_{stat}$. Furthermore, we know that $C(x^0, 0)$ is the only connected

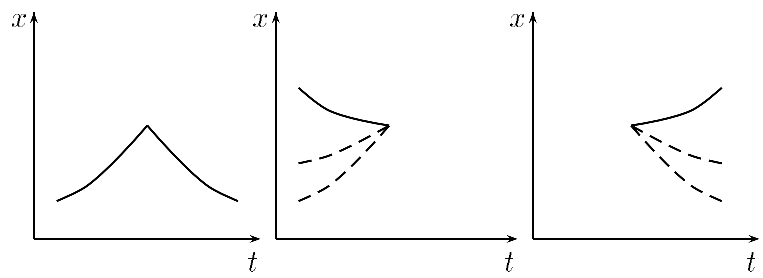


Figure 3.1

component in $\sum_{stat}$ crossing the hyperplane $\{(x,t) \in \mathbb{R}^n \times \mathbb{R} \,|t=0\}$.
We introduce the following condition for $P_2(t)$:

(F1) MFCQ is satisfied for all $x \in M_2(t)$ with $(x,t) \in$ $cl\ C(x^0,0)\ |_{[0,1]}$

Theorem 3.15 *Let $(F,G) \in C^3(\mathbb{R}^n,\mathbb{R})^{s+l}$. Assume (A2) and (A3). Then there exists a PC^2-path in $\sum_{stat}$ connecting $(x^0,0)$ with some point $(x^*,1)$, where x^* is a stationary point of (P) if and only if (F1) is satisfied.* □

Remark concerning the proof: Use the same concept as in the proof of Theorem 2.5.

Remark 3.16 (i) If the condition (F1) is satisfied and if we do not attain $t=1$, then $M(\mu^1)$ is empty, i.e., μ^1 was not a realistic wish of the decision maker. The program package PAFO provides information whether (F1) is satisfied (a)) or not (b)), namely

(a) if there are singularities of the Types 2,3, and 5 (where the MFCQ is satisfied, that means, there is a continuation in $\sum_{stat}$ with the same orientation (cf. Fig. 3. 1(a))).

(b) if there are points of the Types 4 or 5 (where the MFCQ is not fulfilled, that means, the path ends in $\sum_{stat}$ and has a continuation in $\sum_{gc}$ only with the opposite orientation (cf. Fig. 3.1(b) and Example 4.2)).

If there appears a point $(\bar{x},\bar{t})$ of Type 4 when approaching $(\bar{x},\bar{t})$ by local minimizers, then, in Case I, we can jump to another connected component in $\sum_{stat}$ and, in Case II, there is no jump (cf. Fig. 2.4). We have the same situation if a point of Type 5 appears, where the MFCQ is not satisfied. Therefore, using pathfollowing and jumps in the set $\sum_{stat}$ we are not able to give an answer to the question whether $M(\mu^1)$ is empty or not. then we can try to compute connected components in $\sum_{gc}$ and possible jumps there (cf. JUMP II in [12] and further jumps in [14]), but in the worst case we have to find all connected components and we do not know how many of them exist. Consequently, we only know that $M(\mu^1) \neq 0$ in case we attain $t = 1$.

(iii) Summarizing, we can say that the proposed embedding for finding a point $\hat{x} \in M(\mu^1)$ (if $M(\mu^1) \neq 0$) are much better than those used in [7], [8], [9], [10] for instance.

The only disadvantage in comparison to $P_1(t)$ is the fact that the computed goal realizer $x^* \in M(\mu^1)$ is not necessarily a locally efficient point. From this point of view, we discuss other embeddings (cf. e.g. [20]) for trying to overcome this problem in the next article.

4 Illustrating Examples

Example 4.1:

$$\begin{aligned} f_1(x_1, x_2) &= x_1 \\ f_2(x_1, x_2) &= x_2 \\ g_1(x_1, x_2) &= -16x_1^2 + 4x_2^2 + 64 \\ g_2(x_1, x_2) &= 4x_1^2 - 16x_2^2 + 64 \\ g_3(x_1, x_2) &= x_1^2 + x_2^2 - 100 \end{aligned}$$

First, we use the embedding $P_1(t)$ with the following starting situation:

$$\lambda_1^0 = 3, \quad \lambda_2^0 = 1, \quad \mu^0 = (-3, 4.472), \quad \mu^1 = (3, -5), \quad x^0 = (-3, 4.472)$$

Then we obtain the following figures

We do not attain $t = 1$. The last point is the point of Type 5 $x = (-2.309, 2.309)$ at $t = 0.228$. If we decrease t, the feasible set will become empty. Therefore, we use the embedding $P_2(t)$ with the starting point $x^0 = (-3, 4.472), \mu = (3, -5), f^0 = (1, 2), g^0 = (3, 4, 5), p = 200$. Then we obtain the Figures 4.3 and 4.4.
From the starting point we attain $t = 1$ at $x = (-3.202, -5)$, which is obviously a goal realizer for our example. But, between the point of Type 5 (-2.096,2.053) at $t = 0.221$ and the point of Type 4 (-0.918,0.000) we have to follow generalized critical points. At this point of Type 4 it is impossible to jump. The Figures 4.5 and 4.6 show all possible paths in the interval $[-1, 1]$ and we see that we attain $t = 1$ as a stationary point with the above path only. There are points of Type 4 and Type 5 because the EnMFCQ is not satisfied.

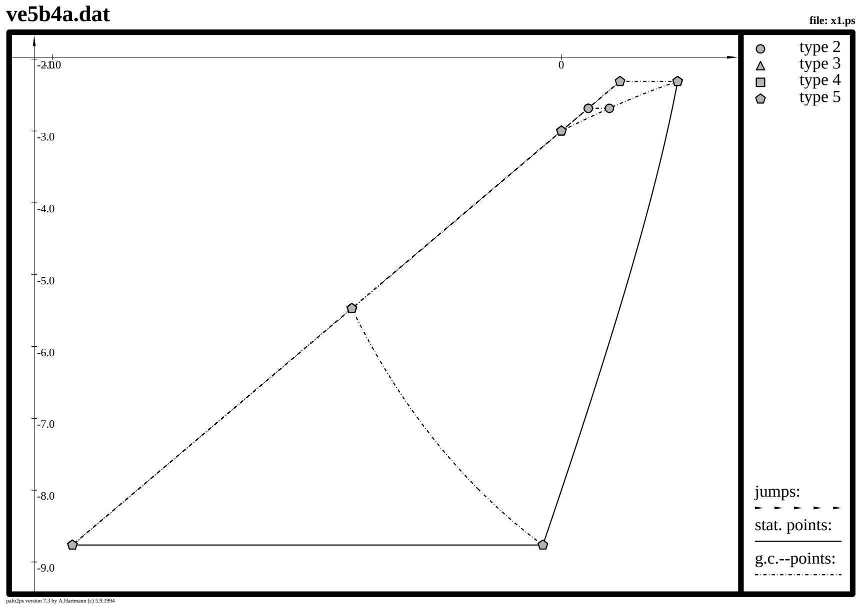


Figure 4.1

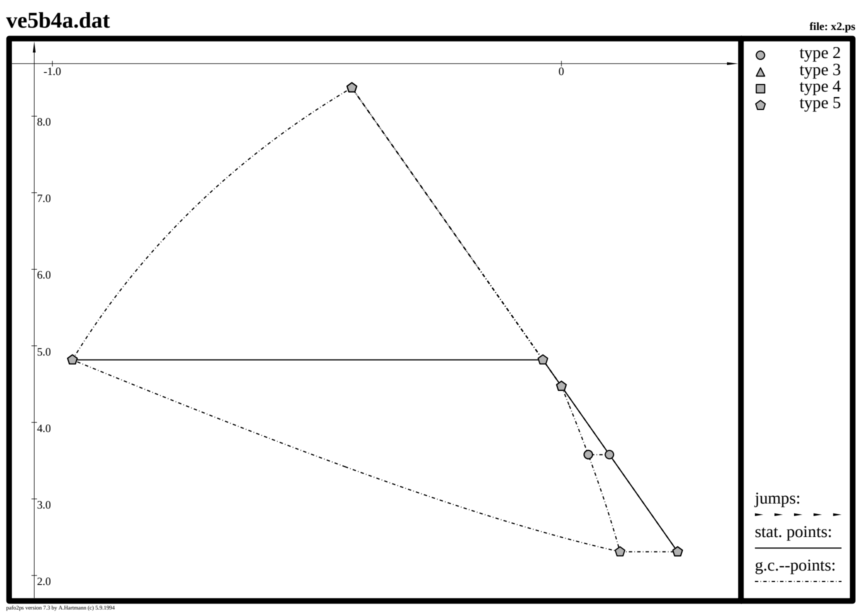


Figure 4.2

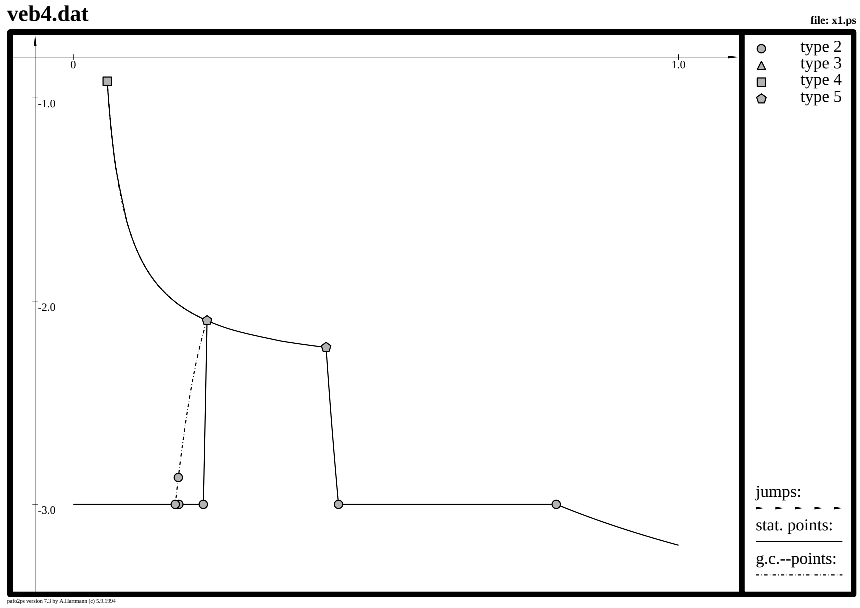


Figure 4.3

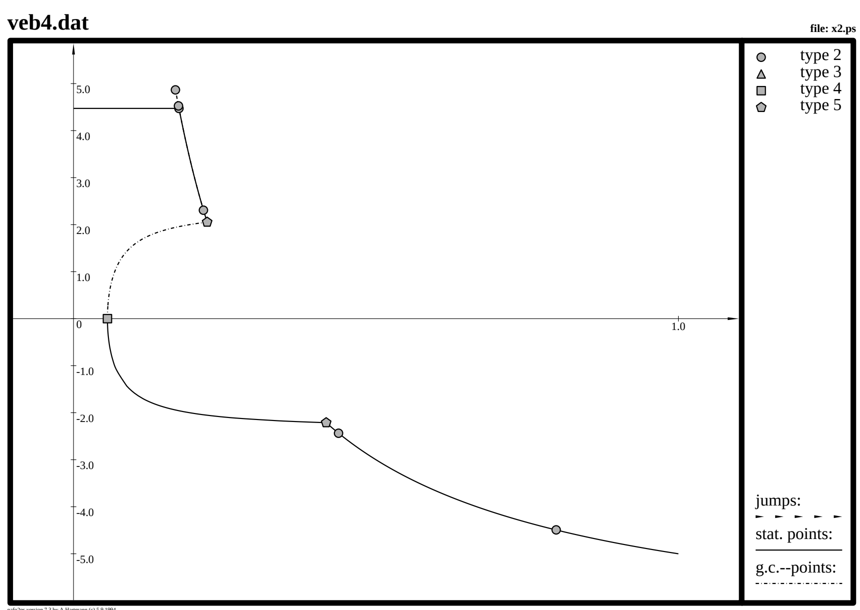


Figure 4.4

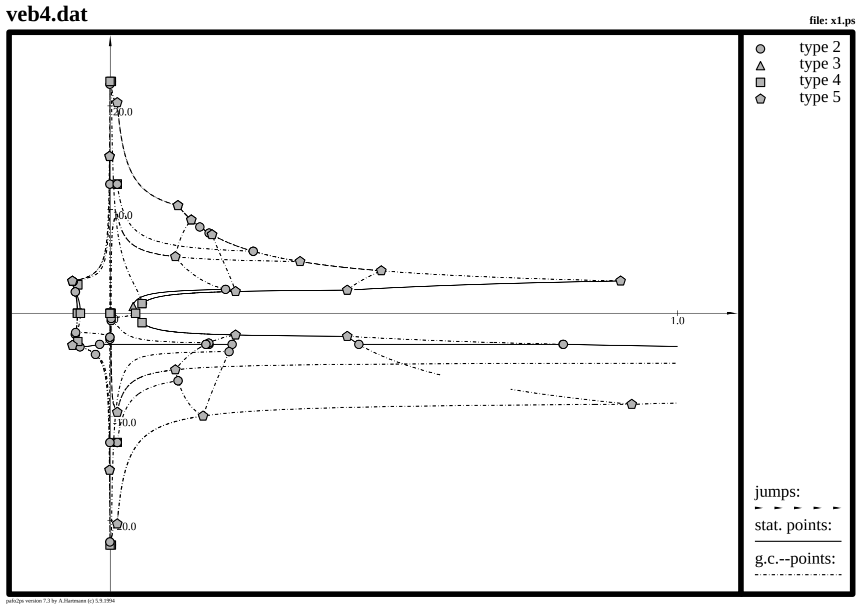


Figure 4.5

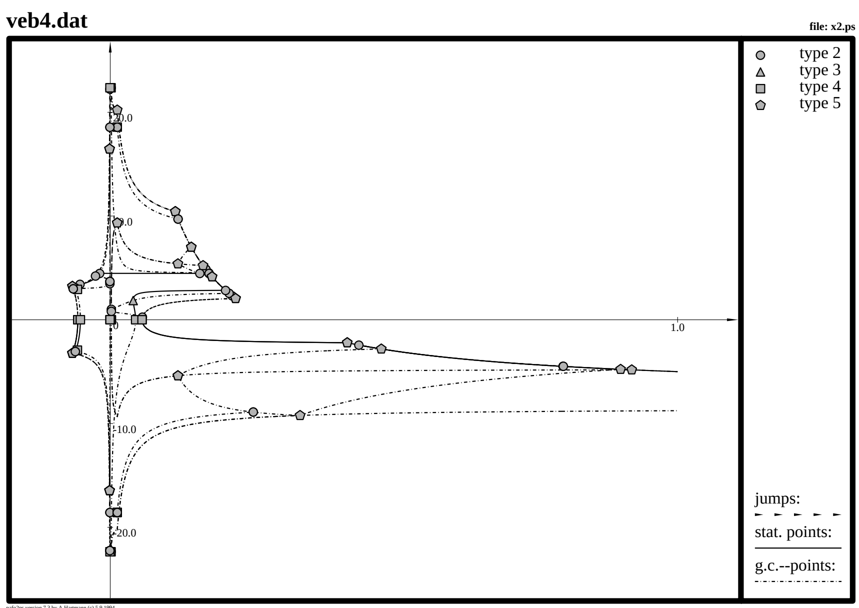


Figure 4.6

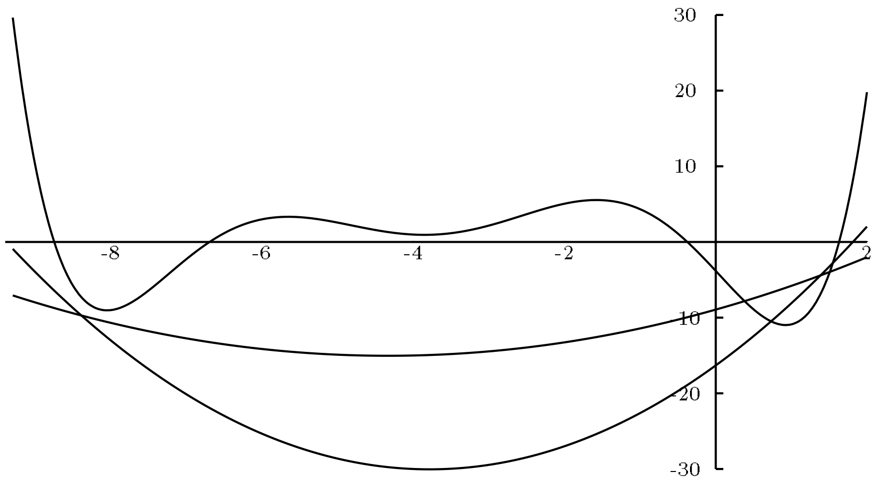


Figure 4.7

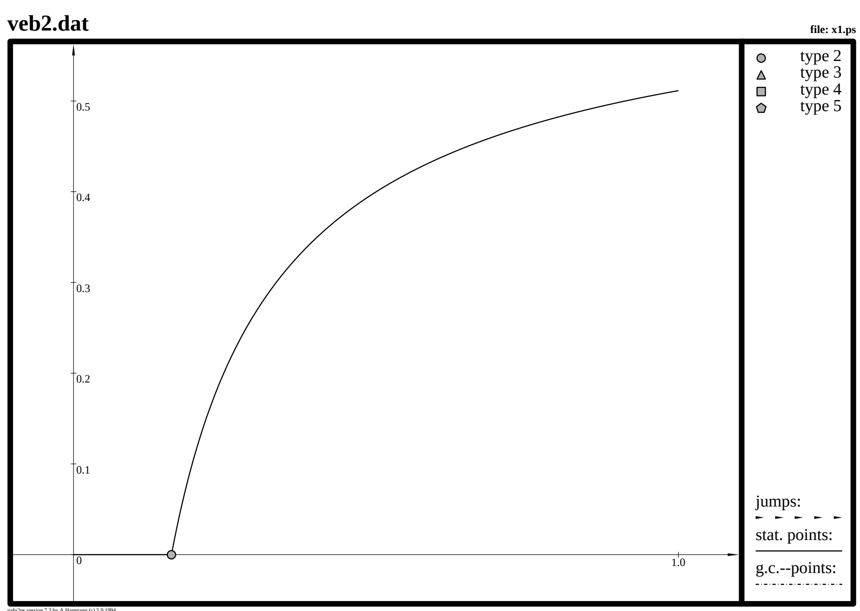


Figure 4.8

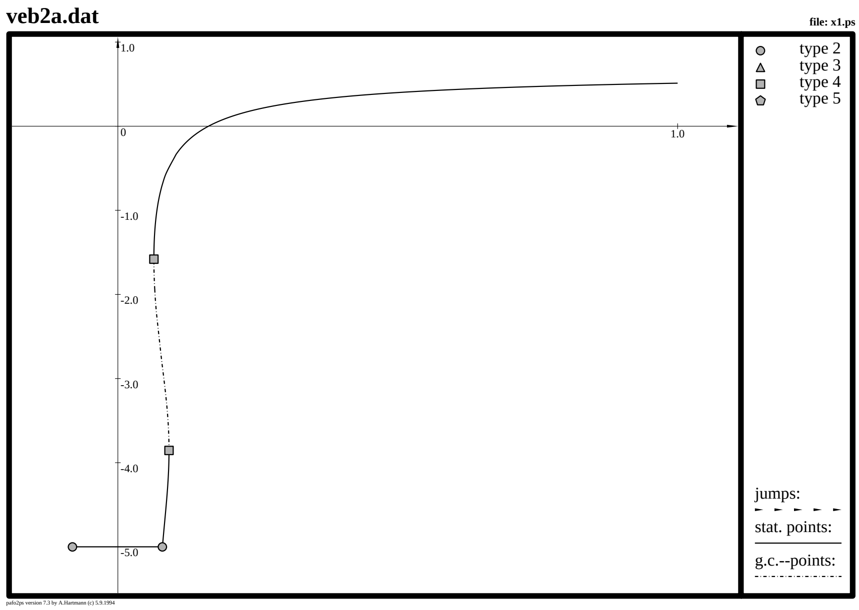


Figure 4.9

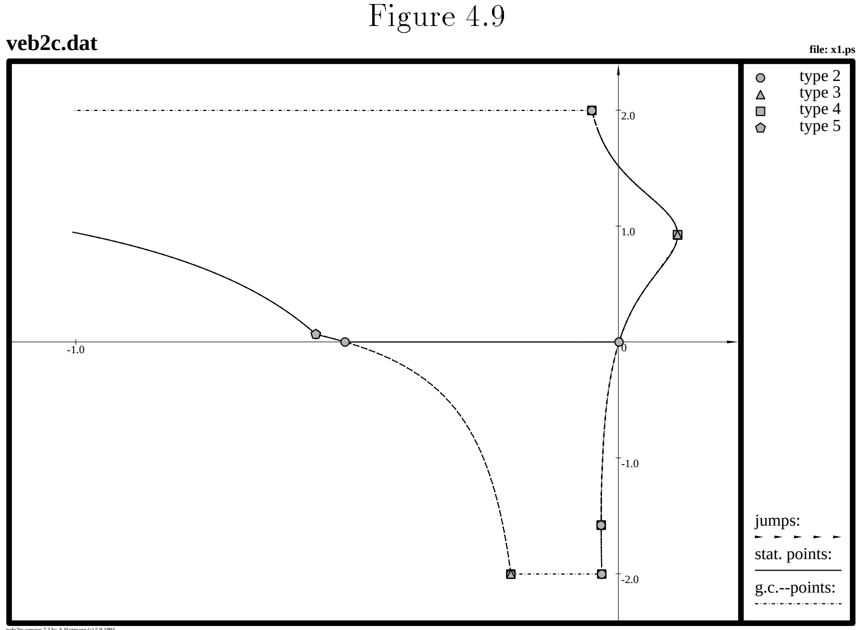


Figure 4.10

Example 4.2:

$$\begin{aligned} f_1 &= 0.007x^6 + 0.153x^5 + 1.138x^4 + 2.999x^3 - 0.602x^2 - 10.761x - 3.824 \\ f_2 &= 0.958x^2 + 7.254x - 16.312 \\ g &= 0.324x^2 + 2.812x - 8.936 \end{aligned}$$

In Figure 4.7 we see these functions. For this example we only use the embedding $P_2(t)$. First, we choose the following starting point:

$$x^0 = 0, \quad \mu^1 = (-9, -2), \quad f_0 = (1, 2), \quad g_0 = 3, \quad p = 4$$

Without any problems we attain the goal realizer $x = 0.511$ at $t = 1$ (see Figure 4.8) ((F1) is satisfied). If we choose the starting point

$$x^0 = -5, \quad \mu^1 = (-9, -2), \quad f_0 = (1, 2), \quad g_0 = 3, \quad p = 36$$

we see that the EnMFCQ is not satisfied. Hence, as we can see in Figure 4.9, we obtain points of Type 4, where we cannot jump and we have to follow the generalized critical points and attain $t = 1$.

Now we take the first starting point only with the new goal $\mu^1 = (-15, -10)$, which is not realistic, obviously. Figure 4.10 shows that we do not attain $t = 1$. For $t = 0.199$ we have a turning point of Type 4 at $x = 0.924$ where it is not possible to jump. If we choose $t > 0.199$, the feasible set is empty.

References

[1] Charnes, A., Cooper, W. (1961): Management Models and Industrial Applications of Linear Programming. New York

[2] Fandom Noubiap, R. (1997) : On Modifications of the Standard Embedding in Nonlinear Optimization, in : Guddat, J., Jongen, H.Th., Nožička, F. Still, G., Twilt, F. (eds.): Parametric Optimization and Related Topics IV, Ser. Approximation and Optimization, Peter Lang Verlag

[3] Gal, T. (1982): Efficient Sets in Vector Maximum Problems - a Brief Survey, in: Hansen, P.(ed.): Essays and Surveys on Multiple Criteria Decision Making, Lecture Notes in Economic and Math. Systems 209, 94-114.

[4] Gfrerer, H., Guddat, J., Wacker, Hj., Zulehner, W. (1985): Pathfollowing Methods for Kuhn-Tucker Curves by an Active Index Set Strategy. In: Bagchi, A., Jongen, H.Th. (eds.) Systems and optimization. Lecture Notes in Control and Information Sciences 66, Springer-Verlag Berlin, Heidelberg, New York, 111-131.

[5] Gollmer, R., Kausmann, U., Nowack, D., Wendler, K. (1997): Computerprogramm PAFO, Humboldt-Universität Berlin, Institut für Mathematik.

[6] Gomez, W. (1997): On Generic Quadratic Penalty Embeddings for Nonlinear Optimization Problems. Humboldt-Universität, Institut für Mathematik, Preprint Nr. 97-18, submitted to Mathematics of Operations Research

[7] Gomez, W.(1997): Vector Optimization: Singularities, Regularizations. To appear in: Mathematical Methods of Operations Research.

[8] Guddat, J., Guerra, F. (1987): Using Pivoting and Continuation Methods, Archivum Automatyki i Telemechaniki, Tom XXXII, 301-318.

[9] Guddat, J., Guerra, F., Tammer, K., Wendler, K. (1985): Multiobjective and Stochastic Optimization Based on Parametric Optimization. Math. Research, Vol. 26.

[10] Guddat, J., Guerra, F., Nowack, D. (1991): Multiobjective Optimization: Singularities, Pathfollowing and Jumps. In: G. Fandel, H. Gehring (eds.), Operations Research, Springer Verlag, 81-108

[11] Guddat, J., Guerra, F., Nowack, D. (1997): On the Role of the Mangasarian-Fromovitz Constraint Qualification for Penalty-, Exact Penalty-, and Lagrange Multiplier Methods. Lectures Notes in Pure and Applied Mathemathics, Vol. 195, A.V. Fiacco (ed.): Mathematical Programming with Data Perturbations, Verlag Marcel Dekker, 159-183.

[12] Guddat, J., Guerra, F., Jongen, H.Th. (1990): Parametric Optimization: Singularities, Pathfollowing and Jumps. BG Teubner, Stuttgart and John Wiley, Chichester.

[13] Guddat, J., Jongen, H.Th., Rückmann, J.-J. (1986): On Stability and Stationary Points in Nonlinear Optimization. J. Austral. Math. Soc., Ser. B 28, 36-56.

[14] Guddat, J., Rückmann, J.-J. (1994): One-Parametric Optimization: Jumps in the Set of Generalized Critical Points. Control and Cybernetics 22, 1/2, 139-151.

[15] Jongen, H.Th., Jonker, P., Twilt, F. (1986): On One-Parametric Families of Optimization Problems. Equality Constraints. JOTA 48, 141-161.

[16] Jongen, H.Th., Jonker, P., Twilt, F. (1986): Critical Sets in Parametric Optimization. Math. Programming 34, 333-353.

[17] Kojima, M., Hirabayashi, R. (1984): Continuous Deformation of Nonlinear Programs. Math. Progr. Study 21, 150-198.

[18] Rückmann, J.-J., Tammer, K. (1992): On Linear-Quadratic Perturbations in One-Parametric Non-Linear Optimization. Systems Science 18 1, 37-48.

[19] Wendler, K. (1993): Implementation of a Pathfollowing Procedure for Solving Nonlinear One-Parametric Optimization Problems. In: Brosowski, B. et al. (eds.) Multicriteria Decision. In: Ser. Approximation and Optimization. Verlag Peter Lang, Frankfurt a.M., Berlin, Bern, New York, Paris, Wien, 139-163.

[20] Wierzbicki, A. P. (1986): On the Completeness and Constructiveness of Parametric Characterization to Optimization Problems, OR Spectrum 81, 73-87